Non-categoricity and Incompleteness: on the Limits of Formal Systems

Constantin C. Brîncuș

Institute of Philosophy and Psychology, Romanian Academy

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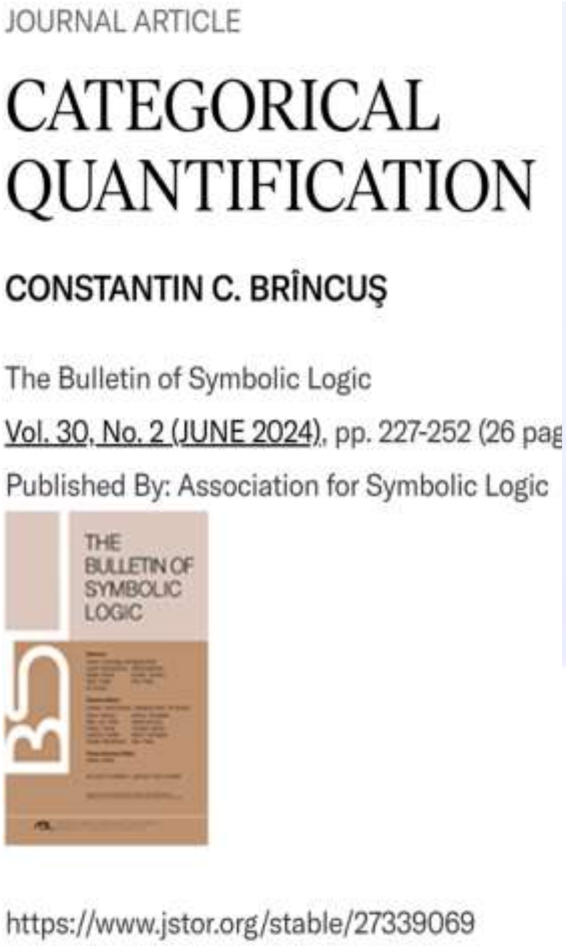
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CATEGORICAL QUANTIFICATION

CONSTANTIN C. BRÎNCUŞ

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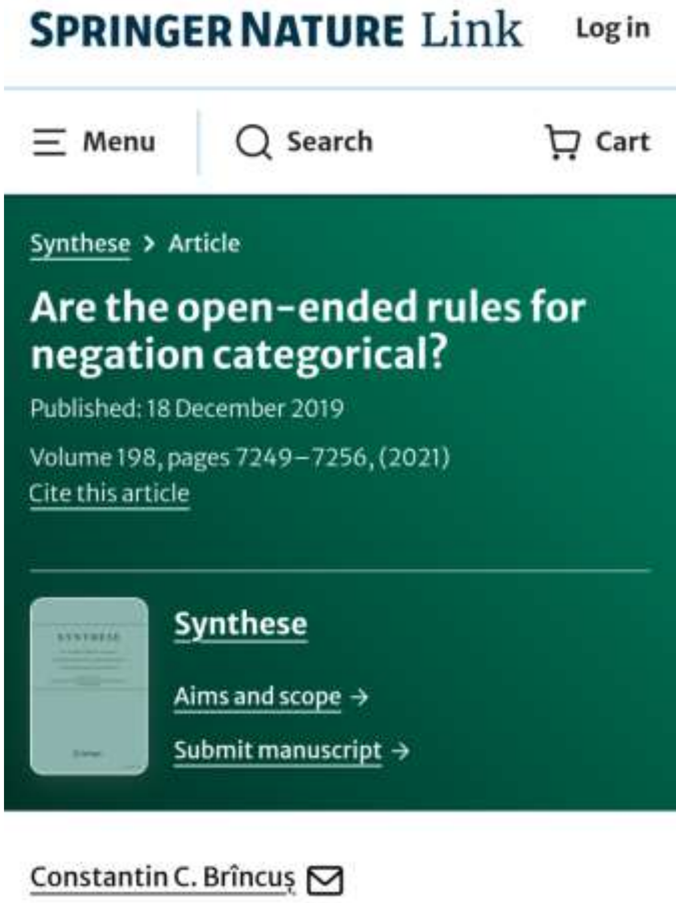
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Published: 18 December 2019

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- "**Inferential Quantification and the ω-Rule**"**.** In Antonio d'Aragona (Ed.) ***Perspectives on Deduction*, Synthese Library Series, Springer, 2024, pp. 345-372**
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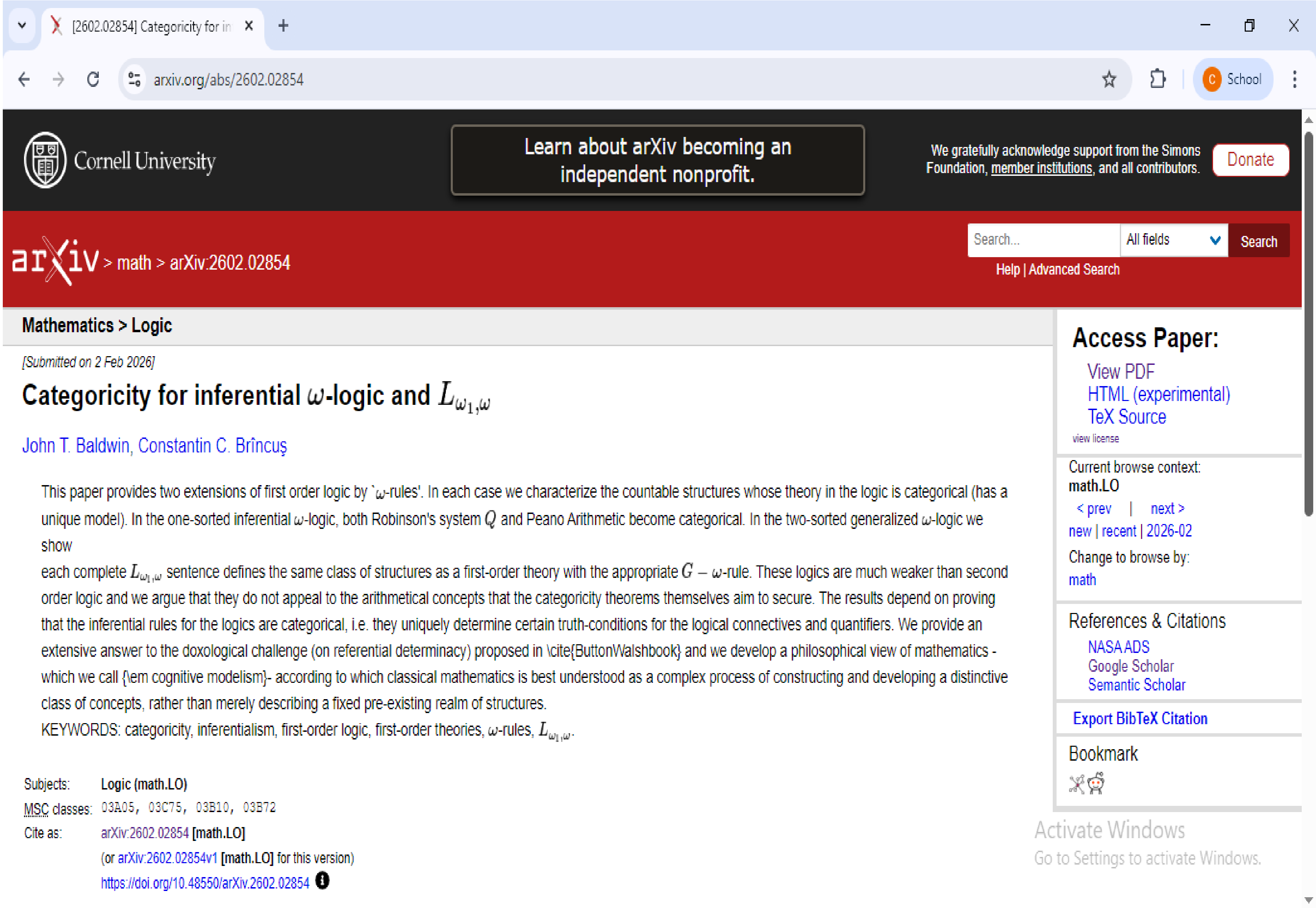

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Mathematics > Logic

[Submitted on 2 Feb 2026]

Categoricity for inferential ω-logic and $L_{\omega_1,\omega}$

John T. Baldwin, Constantin C. Brîncuş

This paper provides two extensions of first order logic by `ω-rules'. In each case we characterize the countable structures whose theory in the logic is categorical (has a unique model). In the one-sorted inferential ω-logic, both Robinson's system Q and Peano Arithmetic become categorical. In the two-sorted generalized ω-logic we show

each complete $L_{\omega_1,\omega}$ sentence defines the same class of structures as a first-order theory with the appropriate $G-\omega$-rule. These logics are much weaker than second order logic and we argue that they do not appeal to the arithmetical concepts that the categoricity theorems themselves aim to secure. The results depend on proving that the inferential rules for the logics are categorical, i.e. they uniquely determine certain truth-conditions for the logical connectives and quantifiers. We provide an extensive answer to the doxological challenge (on referential determinacy) proposed in \cite{ButtonWalshbook} and we develop a philosophical view of mathematics - which we call {\em cognitive modelism}- according to which classical mathematics is best understood as a complex process of constructing and developing a distinctive class of concepts, rather than merely describing a fixed pre-existing realm of structures.
KEYWORDS: categoricity, inferentialism, first-order logic, first-order theories, ω-rules, $L_{\omega_1,\omega}$.

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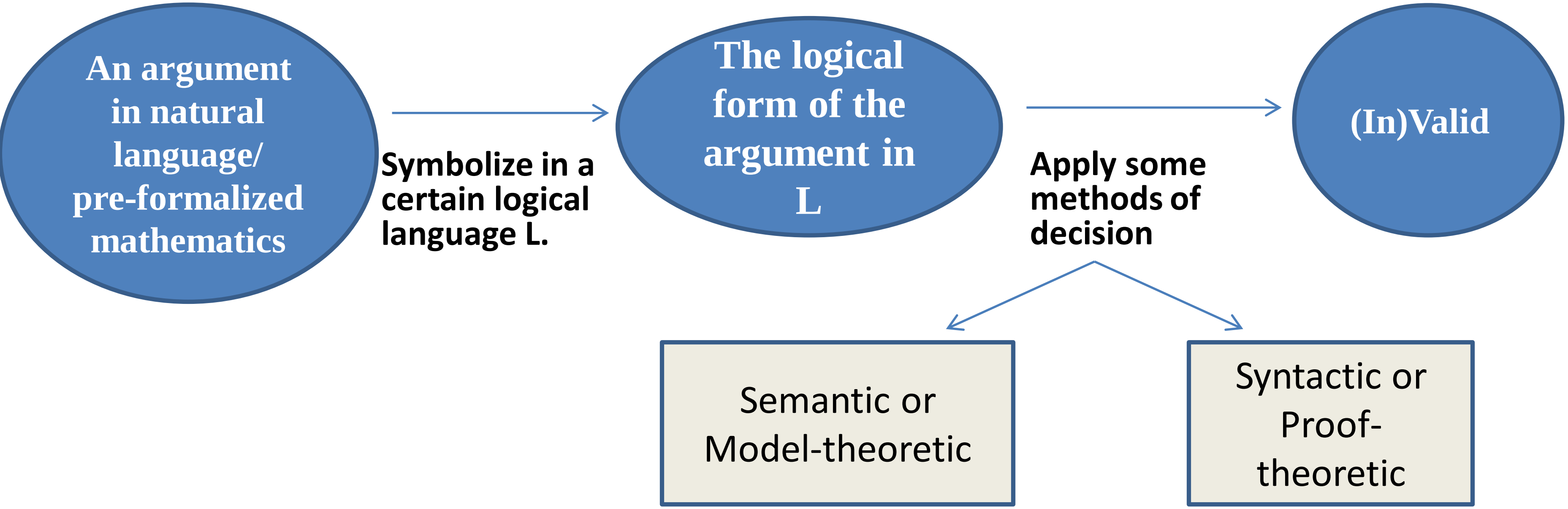
Deductive Logical Reasoning

- Logic is the study of valid arguments.



An argument in natural language/ pre-formalized mathematics

Symbolize in a certain logical language L.

The logical form of the argument in L

Apply some methods of decision

(In)Valid

Semantic or Model-theoretic

Syntactic or Proof-theoretic

Logic:

1. A language L (well formed formula).
2. An interpretation for L/ a model-theory for L (Truth in a model and logical consequence)
3. A deductive system for L/ a proof-theory for L (Sound derivation and proof)

Propositional and First-Order Logic

Language

1. Propositional letters: A, B, C, D …
2. Logical operators: & (and), v (or), ~ (not), → (if …, then), ↔ (if, and only if, …, then ..)
3. Predicate letters: F_, G_, H_ …
4. Individual constants: a, b, c, d …
5. Individual variables: x, y, z …
6. Quantifiers: universal (∀) (for any), existential (∃) (for some/there is at least one)
7. Parantheses: ‘(’, ‘)’

Model Theory

- Each sentence is either true or false.
- Truth-conditional definitions for logical operators and quantifiers.
- Validity: there is no interpretation which makes the premises true and the conclusion false.
- ⊨ model-theoretic logical consequence.

Proof-theory

- A system of axioms or rules of inference that implicitly define the logical operators.
- Validity: the conclusion can be logically derived from the premises by using the axioms or rules of inference.
- ⊢ proof-theoretic logical consequence

Soundness: If P1, P2, …Pn ⊢ C, then P1, P2, …Pn ⊨ C

Semantic Completeness: If P1, P2, …Pn ⊨ C, then P1, P2, …Pn ⊢ C

First-Order Theories: Peano Arithmetic

Axioms of PA:

Ax. 1: $(\forall x)(0 \neq Sx)$

Ax. 2: $(\forall x)(\forall y)(x \neq y \longrightarrow Sx \neq Sy)$

Ax. 3: $(\forall x)(x + 0 = x)$

Ax. 4: $(\forall x)(\forall y)(x + Sy = S(x+y))$

Ax. 5: $(\forall x)(x \bullet 0 = 0)$

Ax. 6: $(\forall x)(\forall y)(x \bullet Sy = x \bullet y + x)$

Induction Principle: $[\varphi(0) \;\&\; (\forall n)(\varphi(n) \longrightarrow \varphi(Sn)] \longrightarrow (\forall x)\varphi(x)$

Model Theory

- ➢We have the notion of model from FOL.
- ➢**PA is the class of models that satisfy the axioms of PA**

Proof-theory

- ➢We have the notion of sound derivation from FOL.
- ➢**PA is the set of deductive consequences of the axioms derived in FOL.**

Outline

I. Two Ideals of Logical Formalization as applied to AS and Logical Systems

II. Logical Inferentialism and the Categoricity Problem

III. Approaches based on Open-ended Logical Rules of Inference and their Semantic or Infinitary Commitments

IV. The Inferential Omega Rule and the Categoricity of PA.

Two Ideals of Logical Formalization

We may distinguishes two main uses of logic in mathematics (Hintikka 1989, 1995):

1. **The descriptive use of logic:**
   - The use of logical notions for the purpose of capturing different structures studied in mathematical theories.
   - **Ideal**: to attain a ***categorical axiomatization*** S for a mathematical theory, i.e., any two models of S are isomorphic. (descriptive completeness)

2. **The deductive use of logic:**
   - The use of logical inferences for systematizing and criticizing mathematicians' reasoning about the structures they are interested in.
   - **Ideal**: to attain a ***deductively complete system*** S, i.e. for each sentence C in the language of S, either $S\vdash C$ or $S\vdash \sim C$.

Two Ideals of Logical Formalization

Remark: ***categoricity*** and ***deductive completeness*** are properties of the mathematical theory formally axiomatized in a certain logical system and not of the underlying logical system itself.

The underlying logical system L of S is ***semantically complete*** iff all valid formulas and all consequences can be derived, or are formally represented, in the deductive system, i.e. if $\Gamma \vDash \varphi$, then $\Gamma \vdash_L \varphi$.

Two Ideals of Logical Formalization

Some connections:

- ➢ If S is deductively complete, then L is semantically complete.
- ➢ If S is deductively complete, then the models of S are elementary equivalent.
- ➢ If S is not deductively complete, then either its logic is not semantically complete, or its models are not elementary equivalent, i.e., S is non-categorical.
- ❑ If S is first order PA, which is not deductively complete, since L is semantically complete, then S is non-categorical.

Gödel's Incompleteness Theorem: If we reach a categorical mathematical theory S (containing elementary arithmetic), then our underlying logic cannot be sematically complete. Thus, we can have categoricity only at the price of semantical completeness of the underlying logic. We can have either the categoricity of S, or the semantical completeness of L, but not both. (See Hintikka 1989, Tennant 2000, Hazen 2006, Smith 2020)

Which notion of completeness should we aim at?

1. The **semantic completeness** of the underlying logic
2. The **deductive completeness** of the formalized theory
3. The **descriptive completeness**, i.e., categoricity, of the formalized theory
4. The **descriptive completeness**, i.e., categoricity, of the underlying logic. (***Absoluteness***)

I. **Warren (2020) argued that all of them are desirable.**

-he preserves the classical FOL and adds open-endedness and the ω-rule; FOL+ ω-rule gives us semantic and deductive completeness and open-endedness is meant to provide us with categoricity.

II. **(Murzi and Topey 2021) argued that 3 and 4 are desirable.**

-they are interested in the categoricity of logic and mathematical theories and, thus, by using open-endedness, they argue for 4, and use SOL to obtain 3.

III. **I also take 1-4 to be desirable** and I argue that the (Inferential) ω-rule help us attaining them (1, 2, 3, 4).

Outline

I. Two Ideals of Logical Formalization as applied to AS and Logic

II. **Logical Inferentialism and the Categoricity Problem**

III. Approaches based on Open-ended Logical Rules of Inference and their Semantic or Infinitary Commitments

IV. The Omega Rule and Inferential Conservativity

(Model Theoretic) Logical Inferentialism and Categoricity

Model-theoretic Inferentialism

1) The meanings of the logical terms are uniquely determined by the rules of inference that govern their use in a logical calculus and
2) These meanings are to be defined in model-theoretic terms (truth-conditions, extension, reference).

Categoricity

A system of rules or axioms L is categorical iff all its models are standard.

or

A calculus in categorical iff it uniquely determines the intended model-theoretic meanings of the logical terms.

Conjunction: PC & NTT

(&I)

$$\frac{p \quad q}{p \,\&\, q}$$

(&E)’

$$\frac{p \,\&\, q}{p}$$

(&E)’’

$$\frac{p \,\&\, q}{q}$$

	p	**q**	**p & q**
C1	⊤	⊤	⊤
C2	⊤	⊥	⊥
C3	⊥	⊤	⊥
C4	⊥	⊥	⊥

(&I) : C1
(&E)’ : C3, C4
(&E)’’ : C2, C4

Disjunction: PC & NTT

(vI)’

$$\frac{p}{p \vee q}$$

(vI)’’

$$\frac{q}{p \vee q}$$

(vE)

$$\frac{p \vee q \qquad \overset{[p]}{r} \qquad \overset{[q]}{r}}{r}$$

	p	q	p v q
D1	⊤	⊤	⊤
D2	⊤	⊥	⊤
D3	⊥	⊤	⊤
D4	⊥	⊥	?

(vI) : D1, D2
(vI)’ : D1, D3
(vE) : -------

Material Implication: PC & NTT

(→I)

$$\dfrac{\begin{array}{c}[p]\\ q\end{array}}{p \to q}$$

(→E)

$$\dfrac{p \quad p \to q}{q}$$

	p	q	p → q
I1	$\top$	$\top$	$\top$
I2	$\top$	$\bot$	$\bot$
I3	$\bot$	$\top$	$\top$
I4	$\bot$	$\bot$	?

(→ I) : I1, I3

(→ E) : I2

Negation: PC & NTT

(~I)

$$\frac{\begin{array}{c}[p]\\ \text{人}\end{array}}{\sim p}$$

(~E)

$$\frac{p \qquad \sim p}{\text{人}}$$

	p	~p
N1	⊤	?
N2	⊥	?

(~I) : …..

(~E): …..

Logical Inferentialism and Categoricity

Universal quantifier rules

Let τ be a first-order vocabulary, ϕ(t) be any sentence that contains the individual constant t, and x a variable in L.

∀I :

$$\frac{\begin{array}{c}\vdots\\ \phi(t)\end{array}}{(\forall x)\phi(x)}$$

∀E :

$$\frac{(\forall x)\phi(x)}{\phi(t)}$$

Restrictions on ∀I: t does not occur in any premise or assumption on which ϕ(t) depends, and x does not occur in ϕ(t). x replaces all and only occurrences of t in ϕ(t).

Logical Inferentialism and Categoricity

Non-categoricity of ∀I and ∀E-rules

(Carnap) Consider a vocabulary that has only two unary predicates F and G and a countable number of individual constants.

1. Let v be a standard truth-theoretic valuation. In particular, it interprets (∀x)F(x) such that v(F(c)) = T for any constant c; ‘the values of v(G(c)) are irrelevant, for any constant c’.
2. Define a valuation v′ that maps the individual constants in the vocabulary onto the objects from a denumerable domain and agrees with v on the quantifier-free formulas. Interpret (∀x)F(x) in v′ as ‘v(F(c)) = T for any constant c and v′(G(c)) = T, for a particular c, namely b’. In v′ the universal quantifier has a richer content than that given by the usual truth conditions in v.

Logical Inferentialism and Categoricity

Non-categoricity of ∀I and ∀E-rules -continued

We consider two options for how v interprets Gb, and thus how v′ interprets (∀x)F(x), relative to the corresponding structure.

a) If v(G(b)) = v′(G(b)) is true, then the ∀E and ∀I rules preserve their soundness under v′ since Gb does not provide a counterexample.
b) If v(G(b)) is false, then ∀E is sound in v′ since its premise, i.e. v′((∀x)F(x)), is false. Likewise, if v(G(b)) is false, then the ∀I-rule preserves its soundness since both F and G are atomic predicates and there is no generalizable or free variable proof for Fc.

Since v′ preserves the soundness of the ∀I and ∀E-rules, v′ is admissible, although it interprets the universal quantifier non-standardly.

Garson’s Non-Categoricity Theorem

- **The objectual interpretation of $\forall$:**

  $v(\forall xFx)=\text{true}$ iff for all d in D, $v(Fd/x)=\text{true}$.
- **The substitutional interpretation of $\forall$:**

  $v(\forall xFx)=\text{true}$ iff for all t in L, $v(Ft/x)=\text{true}$.
- If every object from the domain of quantification D is named in the language, then the two interpretations are equivalent.

Theorem 14.3: $S\forall$ does not express $\|s\forall\|$, nor does it express $\|d\forall\|$.

Proof sketch: The set $\{A^y/x: y \text{ is a variable of } L\} \cup \{\sim\forall xA\}$ is consistent in $S\forall$, and the set *e* of all wffs B such that $\{A^y/x: y \text{ is a variable of } L\} \cup \{\sim\forall xA\} \vdash B$ is deductively closed and so a member of $[S\forall]$. The set *e* however, although it contains A^y/x for each variable y, it does not contain $\forall xA$ on pain of inconsistency.

v^{ω} (A^y/x, for each variable y)=true, $v^{\omega}(\forall xA)=\text{false}$.

Logical Inferentialism and Categoricity

Carnap's Non-Categoricity Results

Propositional Logic

$\mathbf{v}^{\mathbf{T}}(\varphi)$ = true, for all wff φ of L. Thus, $\mathbf{v}^{\mathbf{T}}(A)= \mathbf{v}^{\mathbf{T}}({\sim}A)$ =true.

$\mathbf{v}{\vdash}(\varphi)$ = true, when φ is a theorem

(φ)=false, when φ is not a theorem.

Thus, $\mathbf{v}{\vdash}(A)= \mathbf{v}{\vdash}({\sim}A)$=false, but $\mathbf{v}{\vdash}(A \vee {\sim}A)$= true.

Quantificational Logic

$\mathbf{v}^{+}(\forall xPx)= v^{*}(Pa \;\&\; Pb \;\&\; Pc \;\&\; \ldots \& \; Qb)$=true

$(\exists xPx) = v^{*}(Pa \vee Pb \vee Pc \vee \ldots\ldots \vee {\sim}Qb)$=true

Thus, we have models of propositional and quantificational calculi in which the logical terms have non-standard meanings.

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Warren’s (2020) argument for the categoricity of logic

Theorem: $\mathrm{Ext}(x) = D$ iff $\mathrm{Ext}(x) \in \mathrm{Ext}(\forall)$

(Sufficiency): Let us assume that $\mathrm{Ext}(x) = D$ and, for *reductio*, that $\mathrm{Ext}(x) \notin \mathrm{Ext}(\forall)$. We add the predicate ‘$\varphi$’ to our language such that $\mathrm{Ext}(\varphi) = \mathrm{Ext}(x)$. Let c be an individual constant such that $\mathrm{Ext}(c) = o$, for some member o of D. We are now in an expanded language where “φc” is true for some arbitrary ‘c’, but “$(\forall x)\varphi x$” is false. This contradicts the validity of open-ended validity of the $\forall$-introduction rule. Hence, $\mathrm{Ext}(x) \in \mathrm{Ext}(\forall)$. (Warren 2020: 85-86)

Warren’s (2020) argument for the categoricity of ∀

Theorem: Ext(x) =D iff Ext(x) ∊ Ext(∀)

1	**(1)**	Ext(x) = D	**Premise**
2	**(2)**	Ext(x) ∉ Ext(∀)	**Assumption**
3	**(3)**	Let L’ be L ∪ {φ} such that Ext(φ)= Ext(x)	**Open-endedness**
4	**(4)**	Let c∊ L’, such that ext(c)=o, for some o of D.	**Open-endedness**
1,3,4	**(5)**	“φc” is true in L’ for some arbitrary c.	**1,3,4 Definition**
1,3,4	**(6)**	“(∀x)φx” is true.	**5 ∀I**
2,3	**(7)**	“(∀x)φx” is false.	**2,3 Definition**
1,2,3,4	**(8)**	⋏	**6,7 E~**
1,3,4	**(9)**	~ (Ext(x) ∉ Ext(∀))	**2,8 I~**
1, 3,4	**(10)**	Ext(x) ∊ Ext(∀)	**9 , Definition**

Warren’s (2020) argument for the categoricity of logic

Observation:

- Which is the main reason for which “φc” is true for arbitrary c and, thus, justifies the application of the ∀I-rule? Is it just the fact that we simply stipulate it to be so?
- At a closer inspection we see that the reason for which “φc is true for some arbitrary c” is that we have assumed from the very beginning that φ holds for any object from the domain and c is introduced for an object from D.
- However, this assumption is equivalent with asserting that φ expresses a property shared by all objects in the domain (maybe a logical or mathematical property). This assumption, however, is not part of the general inferential use of the first-order quantifiers in logical and mathematical reasoning.
- In addition, it is not part of the inferentialist idea to know in advance that the formula whose universal closure is to be inferred has the entire domain as its extension.

Murzi and Topey 2021 –*Categoricity by Convention*

Weakened First Order Thesis. The rules of FOL are locally valid with respect to a class of valuations V only if all $v \in V$ are such that, for any ϕ, $\forall x\phi$ is true in v iff $Ext_v(x) \subseteq Ext_v(\phi)$.

Proof: (Necessity) Supose the first-order rules are satisfaction-preserving in v, and let ϕ be any formula with at most x free. First, suppose every object in the range of x in v is in $Ext_v(\phi)$. Then v satisfies$_s$ $\vdash\phi$ for any variable assignment s, in which case v satisfies $\vdash\phi$. So, since $\forall$I is satisfaction-preserving, v satisfies $\vdash\forall x\phi$ as well –i.e. $\forall x\phi$ is true in v.

1	(1) $\forall$I-rule is satisfaction preserving.	**Premise (Local Validity)**
2	(2) $Ext_v(x) \subseteq Ext_v(\varphi)$,	**Premise**
2	(3) v satisfies $\vdash\varphi$	**2 Definition**
1, 2	(4) v satisfies $\vdash\forall x\varphi$, i.e., $\forall x\varphi$ is true in v	**1, 3 $\forall$I**

Murzi and Topey 2021 –*Categoricity by Convention*

Observation I

- We emphasize from the very beginning that the $\forall$I-rule is meant to be a first-order rule and, thus, it has to be a finitary rule, i.e., if a conclusion φ is derivable from a set of premises Γ, then it has to be derivable from a finite subset Γ' of Γ.
- Now, if we take for granted the local validity of the $\forall$I-rule, then we know that if "$\Gamma \vdash \varphi$" is satisfied by a valuation v, then "$\Gamma \vdash \forall x \varphi x$" will also be satisfied. In particular, the valuation v^{ω} which assigns $\top$ to each member of $\{\varphi^{t}/x\text{: } t \text{ is a term of } L\}$, also has to assign $\top$ to $(\forall x)\varphi x$ in order to preserve the local validity of the rule.
- But this means that the set $\{\varphi^{t}/x\text{: } t \text{ is a term of } L\} \cup \{\sim(\forall x)\varphi x\}$ is inconsistent in Murzi and Topey's formalization of quantificational logic. If this is so, however, we have to acknowledge the presence of an infinitary rule of inference, i.e., of the ω-rule, which guarantees the inconsistency of this set. In other words, $\forall$I-rule is implicitly taken to be an infinitary rule.

Murzi and Topey 2021 –*Categoricity by Convention*

Observation II

- In their proof of the weakened thesis, Murzi and Topey (2021: 3407) use a restricted form of the ∀I-rule:

$$\frac{\vdash \varphi x}{\vdash (\forall x)\varphi x}$$

- Thus, if $(\forall x)\varphi x$ is taken to be a Gödelian sentence of Goldbach type, the sentence will be true, but to derive it from φ one needs to assume the applicability of an infinitary rule of inference. Hence, the local validity of the ∀I-rule makes this rule generally sound and blocks the non-standard valuations v^{+} and v^{ω} if and only if we take it to have an infinitary nature.

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Logical Inferentialism and Infinitary Rules

Why the logical inferentialists should accept the infinite rules for the first order quantifiers?

As.1 Logical inferentialism maintains that our use of the logical expressions in inferences is what determines their meanings.

As.2 Our use of the expressions “all” and “there is” in mathematical inferences leads us beyond the intuitive and finite reasoning.

P1. If human beings do sometimes use infinite rules of inference in their reasoning, then a logical inferentialist should in principle accept the infinite rules for the quantifiers.

P2. Human beings do sometimes use infinite rules of inference in their reasoning.

C. Therefore, a logical inferentialist should in principle accept the infinite rules for the quantifiers.

Logical Inferentialism and Infinitary Rules

Theorem: The result of adding the ω-rule in the standard formalizations of logic, i.e. L^ω, is categorical, i.e., all valuations v from $L^\omega(V)$ are standard.

Lemma: A valuation v is standard if and only if:

(1) $v(\forall x \varphi x)$ is true if and only if for all terms t in Term $v(\varphi^t/x)$

(2) $v(\forall x \varphi x) = \top$ iff for all object d in D, $v(\varphi^d/x) = \top$

Proof:

(Suficiency) Let us assume that $v(\forall x \varphi x)$ is true and let us assume, in addition, that every object in the domain is nameable. By the $\forall$E-rule, every instance of $(\forall x)\varphi x$ is true, i.e. $v(\varphi^t/x)$ is true for all terms t in Term. As a consequence, every object from the domain will be in the extension of the universal quantifier.

(Necessity) Let us assume that $v(\varphi^t/x)$ is true for each term t in Term. The, by the ω-rule, $v(\forall x \varphi x)$ will also be true.

Logical Inferentialism and Infinitary Rules

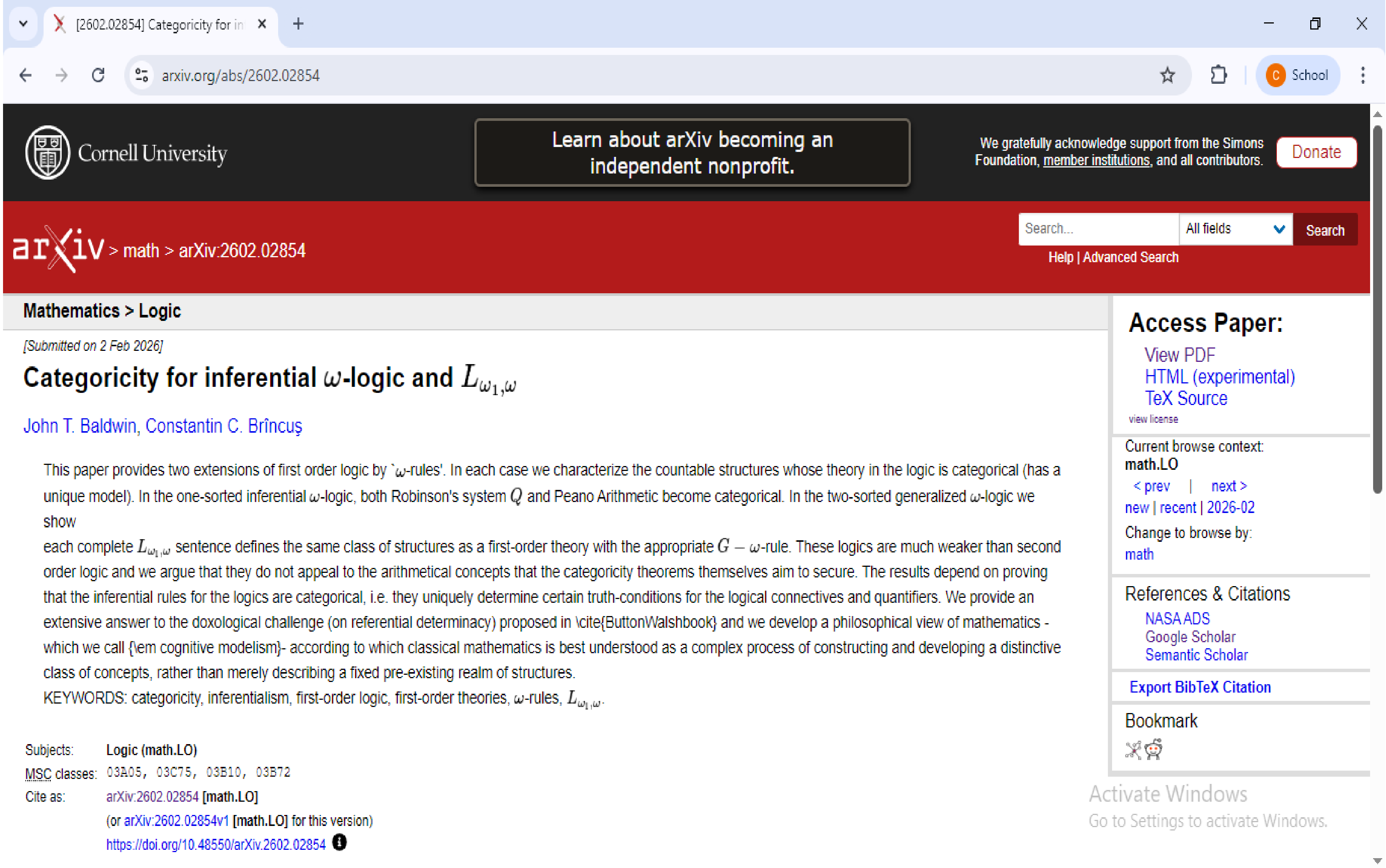
I join Fraenkel, Bar-Hillel and Levy’s (1973: 286) stance in considering Church’s criticism of the use of non-effective rules of inference as not suitable for the purposes of communication (or inferring) as being not very convincing, since:

> Communication may be impaired by this non-effectiveness but is not destroyed. Understanding a language is not an all-or-none affair. Our quite efficient use of ordinary language shows that a sufficient degree of understanding can be obtained in spite of the fact that “meaningfulness”, relative to ordinary language, is certainly not effective.

Communication – Inferring
Understanding – Deriving consequences
Meaningfulness – Being a Consequence

- Actual inferring process may impaired by non-effectiveness, but not destroyed.
- Our efficient use of formal language shows that certain consequences could be derived only by using non-effective rules of inference.

Logical Inferentialism and Infinitary Rules



Mathematics > Logic

[Submitted on 2 Feb 2026]

Categoricity for inferential ω-logic and $L_{\omega_1,\omega}$

John T. Baldwin, Constantin C. Brîncuş

This paper provides two extensions of first order logic by `ω-rules'. In each case we characterize the countable structures whose theory in the logic is categorical (has a unique model). In the one-sorted inferential ω-logic, both Robinson's system Q and Peano Arithmetic become categorical. In the two-sorted generalized ω-logic we show
each complete $L_{\omega_1,\omega}$ sentence defines the same class of structures as a first-order theory with the appropriate $G-\omega$-rule. These logics are much weaker than second order logic and we argue that they do not appeal to the arithmetical concepts that the categoricity theorems themselves aim to secure. The results depend on proving that the inferential rules for the logics are categorical, i.e. they uniquely determine certain truth-conditions for the logical connectives and quantifiers. We provide an extensive answer to the doxological challenge (on referential determinacy) proposed in \cite{ButtonWalshbook} and we develop a philosophical view of mathematics - which we call {\em cognitive modelism}- according to which classical mathematics is best understood as a complex process of constructing and developing a distinctive class of concepts, rather than merely describing a fixed pre-existing realm of structures.
KEYWORDS: categoricity, inferentialism, first-order logic, first-order theories, ω-rules, $L_{\omega_1,\omega}$.

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Categoricity by Inferential ω-rule

The Inferential ω-rule we now introduce is considerably stronger than the usual versions. While the instantiations that must be satisfied in the hypothesis of the ω-rule are the same as usual, the formula ϕ now may have parameters d from $\mathrm{Const}(\tau\ (C)) - \mathrm{Const}(\tau\ )$:

I − ω-rule :

$$\frac{\mathrm{V}\ \{\phi(c, \mathbf{d}) : c \in \mathrm{Const}(\tau\ )\}}{(\forall x)\phi(x, \mathbf{d})}$$

I − ∀E :

$$\frac{(\forall x)\phi(x, \mathbf{d})}{\phi(c, \mathbf{d}),\ \text{for each } c \in C.}$$

Categoricity by Inferential ω-rule

Categoricity of PA –Proof

- Consider the formula $\phi(x, d) : x \neq d$. Thus, if v_M is admissible, then it satisfies the following instance of the inferential ω-rule.

$$\textbf{I} - \boldsymbol{\omega}\textbf{-rule :} \quad \frac{\{\phi(c, d) : c \in \mathrm{Const}(\tau_N)\}}{(\forall x)\phi(x, d)}$$

- Again, if v_M is admissible, it also satisfies the following instance of the inferential ∀E-rule:

$$\textbf{I} - \boldsymbol{\forall}\textbf{E :} \quad \frac{(\forall x)\phi(x, d)}{\phi(e, d), \text{ for each } e \in \mathrm{Const}(\tau_M)}$$

- By this instance of the I-∀E, we have $\phi(d, d)$. But this is a contradiction since $d = d$. Thus v_M violates the rules for the universal quantifier in ω-logic and is not admissible.

Categoricity by Inferential ω-rule

Categoricity of PA –Proof continued

- Note that in any model of Q, the substructure generated from 0 is isomorphic to N = ⟨N, 0,+, ×⟩. That is, N is an algebraically prime model of Q. Thus, an arbitrary countable non-standard model M of Q extends (an isomorphic copy of) N.
- Const(τ_N) is a countable set of constants with arbitrary member denoted by c; in particular v_N maps Const(τ_N) on N. Let v_M be an extension of v_N which enumerates the remaining elements of the τ_M-structure M by constants e ∈ Const(τ_M) where Const(τ_M) = Const(τ_N) ∪ D and D is a countable set disjoint from Const(τ_N). Fix a particular d ∈ D.

Outline

I. **Two Ideals of Logical Formalization as applied to AS and Logical Systems**

II. **Logical Inferentialism and the Categoricity Problem**

III. **Approaches based on Open-ended Logical Rules of Inference and their Semantic or Infinitary Commitments**

IV. **The Inferential Omega Rule and the Categoricity of PA.**

Thank you for your attention!